Italy
ITAMO
2008

Day 1

1 Let $ABCDEFGHILMN$ be a regular dodecagon, let P be the intersection point of the diagonals AF and DH. Let S be the circle which passes through A and H, and which has the same radius of the circumcircle of the dodecagon, but is different from the circumcircle of the dodecagon. Prove that: 1. P lies on S 2. the center of S lies on the diagonal HN 3. the length of PE equals the length of the side of the dodecagon

2 A square $(n-1) \times (n-1)$ is divided into $(n-1)^2$ unit squares in the usual manner. Each of the n^2 vertices of these squares is to be coloured red or blue. Find the number of different colourings such that each unit square has exactly two red vertices. (Two colouring schemse are regarded as different if at least one vertex is coloured differently in the two schemes.)

3 Find all functions $f : Z \to R$ that verify the folowing two conditions: (i) for each pair of integers (m, n) with $m < n$ one has $f(m) < f(n)$; (ii) for each pair of integers (m, n) there exists an integer k such that $f(m) - f(n) = f(k)$.

Day 2

1. Find all triples (a, b, c) of positive integers such that $a^2 + 2^{b+1} = 3^c$.

2. Let ABC be a triangle, all of whose angles are greater than 45° and smaller than 90°. (a) Prove that one can fit three squares inside ABC in such a way that: (i) the three squares are equal (ii) the three squares have common vertex K inside the triangle (iii) any two squares have no common point but K (iv) each square has two opposite vertices onthe boundary of ABC, while all the other points of the square are inside ABC. (b) Let P be the center of the square which has AB as a side and is outside ABC. Let r_C be the line symmetric to CK with respect to the bisector of $\angle BCA$. Prove that P lies on r_C.

3. Francesca and Giorgia play the following game. On a table there are initially coins piled up in some stacks, possibly in different numbers in each stack, but with at least one coin. In turn, each player chooses exactly one move between the following: (i) she chooses a stack that has an even non-zero number of coins $2k$ and breaks it into two identical stacks of coins, i.e. each stack contains k coins; (ii) she removes from the table the stacks with coins in an odd number, i.e. all such in odd number, not just those with a specific odd number. At each turn, a player necessarily moves: if one choice is not available, the she must take the other. Francesca moves first. The one who removes the last coin from the table wins. 1. If initially there is only one stack of coins on the table, and this stack contains 2008^{2008} coins, which of the players has a winning strategy? 2. For which initial configurations of stacks of coins does Francesca have a winning strategy?