Problem of the Day for MATHCOUNTS

CatalystOfNostalgia [13375P34K43V312], compiled by Happyme

March 28, 2008

Problem 1 (HS Math Team Tryout):

If $\sqrt{1 \cdot 9 \cdot 9 \cdot 1} = z^2$, then find all possible real values of z

Problem 2:

A two-digit number AB, where A and B are its digits, is equal to 5 less than three times the sum of its digits. What is the number?

Problem 3 (Mock MATHCOUNTS Test #1):

$\triangle ABC$ has $AB = 13$,$BC = 14$,$CA = 15$. A',B',C' are on BC, CA, AB, respectively, such that $AA' \perp BC$, $BB' \perp CA$, $CC' \perp AB$. Find $AA' + BB' + CC'$, expressed as a decimal rounded to the nearest tenth.

Problem 4 (1995 Madelbrot):

Find the area in the plane contained by the graph of $|x + y| + |x - y| \leq 4$.

Problem 5 (MS Math Team Tryout):

How many ten-element sets of positive integers are there with median 7.5, modes 3 and 6, mean 7.8, and no 8s?

Problem 6:

A unit regular hexagon $ABCDEF$ is drawn in the plane. What is the area of $\triangle ACE$? Express your answer as a common fraction in simplest radical form.

Problem 7:

a, b, c, d, and e are real numbers such that:

$$
\begin{align}
2a + 3b - 4c + d - 3e &= 6 \tag{1}\\
5a + b + 5c + d - 2e &= 9 \tag{2}\\
a - b + 3c + d + e &= 10 \tag{3}\\
9a + 3b + 2c + d + 4e &= 17 \tag{4}\\
-6a + 5b + 5c - 4d &= 1 \tag{5}
\end{align}
$$

Find the value of $a + b + c$.

Problem 8:

In how many ways can Al, Bob, Carl, Dan, Ed, Fred, Greg, Hans, and Ian form three groups of three, each with a leader?

Problem 9 (GBML I Round 4 #3):

Factor completely over the integers, or state that it is unfactorable: $x^2 - zx - 4xy + yz + 3y^2$.

Problem 10 (iTest 2007):

Find the largest number equal to the cube of the sum of its digits.

Problem 11:

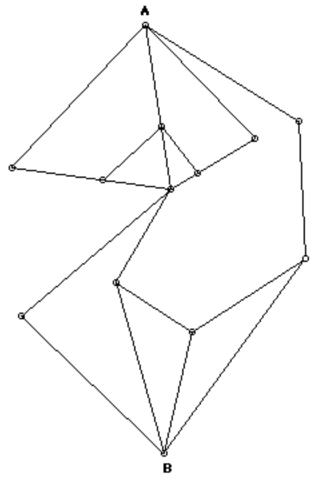
Square $ABCD$ has $AB = 16$. The midpoints of the four sides are constructed, and connected to form a second square. This process is repeated eight times. What is the sum of the areas of the resulting squares?

Problem 12 (Lian's Mock MATHCOUTNS #1):

2007 fair coins are flipped. What is the probability that an odd number of flips result in heads?

Problem 13:

In the diagram given, how many ways are there to travel from A to B only using the given lines?



Problem 14:

Find all real solutions x to the equation:

$(x^2 + 4x - 1)^2 - 5x(x + 4) = 19$ Express your answers as common fractions in simplest radical form if necessary.

Problem 15:

$\triangle ABC$ is equilateral. D, E, F are chosen on BC, CA, AB, respectively, such that $AF = BD = EC$. $DE = x$, $BC = y$. Find the area of $\triangle DEC$, in terms of x and y.

Problem 16:

50 Bucks, Yoya, Mnoyd, and Old Moose all have spent varying amounts of times in jail. 50 Bucks and Yoya have spent a total of 504 days in jail between them. 50 Bucks and Old Moose have spent a total of 801 days in jail between them. Old Moose and Mnoyd have spent a total of 1104 days in jail between them. How many days in jail have Mnoyd and Yoya spent in jail between them?

Problem 17:

Joe writes down the numbers 1 through 20, inclusive, on small slips of paper, then puts them in a hat. If he randomly draws all of the slips of paper, two at a time, what is the probability that 1 and 20 end up in the same pair?

Problem 18 **(Omitted):**

Find all ordered pairs of integers (x, y) such that $x^2 - 4xy + 3y^2$ is prime.

Problem 19 **(MAML** 2007**):**

Find the shortest distance between the line $x + y = 3$ and the curve $(x - 2007)^2 + (y + 2007)^2 = 1$.

Problem 20 **(MAML** 2004**):**

$\triangle ABC$ has D and E on BC such that $AB = 2, AD = 3, AE = 4, AC = 5$ and $\angle BAD = \angle DAE = \angle EAC$. Find the value of $\dfrac{BD}{EC}$.

Problem 21**:**

How many whole numbers are there between $\sqrt[3]{17}$ and $\sqrt{1337}$?

Problem 22**:**

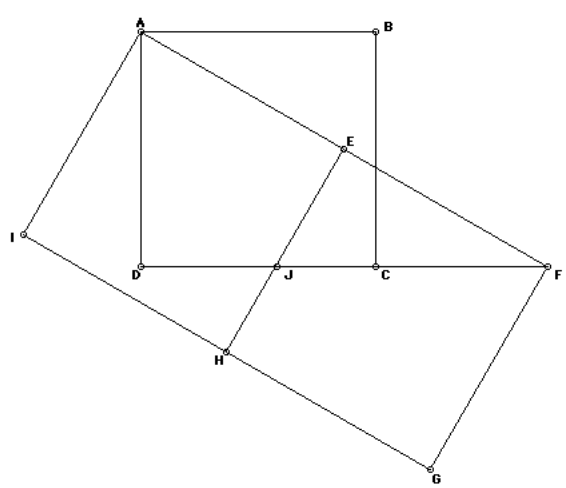
Find the number of ordered pairs of integers (x, y) such that $7x+4y = 2007$ and $-1000 \le x \le 1000$.

Problem 23**:**

How many factors does 118800 have that are either multiples of 2 or multiples of 3?

Problem 24 **(frost13):**

$ABCD$, $AEHI$, $EFGH$ are congruent squares. J and C are on line DF. Find $\angle EJD$.



Problem 25 (GBML 2005):

$\triangle ABC$ is a triangle with D on AB and E on AC, and $\angle DBE = \angle DCE$. $3AE = EC$, $AD = AE$, and $DB = 6$. In addition, $\dfrac{BC}{2} + 6 = AB$. Find the area of the triangle.

Problem 26:

Let a be the remainder when 6^{2007} is divided by 7, let b be the remainder when 2^{2007} is divided by 10, and let c be the remainder when $5^{12345678987654321}$ is divided by 1000. Find the value of $ab + c$.

Problem 27:

We have $A = (0, 0)$ and $B = (6, 0)$. Choose an arbitrary point C on the curve $(x-2007)^2+(y-8)^2 = 20$. Find the minimum possble area of $\triangle ABC$.

Problem 28:

x and y are positive integers such that $x > y$ and $\frac{2}{x} + \frac{3}{y} = \frac{1}{2}$. Find the sum of all possible values of xy.

Problem 29 (Mandelbrot):

In a unit square $ABCD$, a point P is chosen in the interior. What is the probability that $m\angle APB > 120°$?

Problem 30 (AoPS: Intermediate Number Theory):

A triangle with sides 3, 4, 5 make a triangle with an area that is an integer. The next highest triple of consecutive positive integers that makes a triangle with an area that is an integer is 13, 14, 15. Find the third highest such triple.

Problem 31:

Compute the number of terms in the following sequence: $-30.5, -27.5, ..., 998.5$.

Problem 32:

Consider the function $f(a, b, c) = a(1 + \frac{b}{c})$ for positive integral a, b, c. If $a, b, c \leq 10$, find the sum of the minimum value of $f(a, b, c)$ and the maximum value of $f(a, b, c)$.

Challenge

As an extra (difficult) challenge, if a, b, c are arbitary positive numbers, try to find the range of $f(a,b,c)+f(b,c,a)+f(c,a,b)$, or $\sum_{\text{cyc}} a(1+\frac{b}{c})$

Problem 33 (GBML):

The units' digit of $7^{531}-3^{190}$ is multiplied by the units' digit of $7^{531}3^{190}$, and the result is raised to the 25^{th} power, and divided by 5, with remainder r. Find r.

Problem 34 (Simplified Putnam):

$P(x)$ is a polynomial such that $P(P(x)) = 6x - P(x)$. Find all possible values of $P(2007)$.

Challenge

The Actual Putnam Problem: Show that there exists a unique function that maps the postive reals to the positive reals (not necessariliy a polynomial function) $f(x)$ such that $f(f(x)) = 6x - f(x)$.

Problem 35 (A Course in Geometry: Plane and Solid):

$ABCD$ is a parallelogram in the plane, and line PR is in the plane, not intersecting $ABCD$. We have $AP||CR$. Let the feet of the perpendiculars from D and B to PR be S and Q. If $AP = 12, DS = 16, CR = 10, PS = 5, SQ = 2$, find the area of $ABCD$.

Problem 36:

A circle has area 36π, and a chord with length 8 is drawn in the circle. Jamal wants to walk from the center of the circle to the chord. What is the least possible distance he can walk?

Problem 37:

$A = (0,0), B = (15,0)$. $P_k = (k,6)$ for all $1 \le k \le 14$. Find the sum of the areas of all triangles $\triangle ABP_k$ for $1 \le k \le 14$.

Problem 38:

For what values of $k = 1000...0001$, with $2n$ of zeroes, for non-negative n is k prime? If there are infinitely many k, describe them as best you can (e.g. all primes, all perfect squares, all $k \equiv 2 \pmod{13}$, etc.).

Problem 39:

x is real number such that $x + \frac{1}{x} = 4$. Compute the value of $\dfrac{x^{10}+1}{x^5}$.

Problem 40 (PuMAC 2007):

Josh is getting ready for the Sprint Round in his Mathcounts Practice. He chooses a Sprint Round, and before he can make copies, he has to leave his classroom to answer a phone call. While he is gone, Carl enters his room and cuts up the Sprint Round so that all 30 questions are on seperate slips of paper. Carl takes these slips of paper and puts each in its own envelope, and all of the envelopes look the same. Josh comes back, and remembers he made 6 typos in the problems, and has to fix them before making copies. He finds the envelopes lying on the floor, and infuriated, he looks through them one-by-one to find the problems he is looking for, to fix the typos. Find the expected value of the number of envelopes he has to look through before finding the 6 problems he needs to change.

Problem 41:

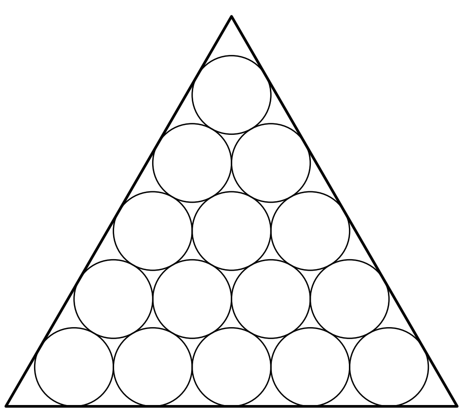
I flip a coin 5 times. What is the probability that I get between 2 and 4 heads, inclusive?

Problem 42 (ARML Super Relay):

x, y, z are real numbers such that $x + y + z = 0, x + y + z^2 = 6$. Find all possible values of $x + y$

Problem 43 (mamathbug):

15 unit circles are put in a "bowling pin" arrangement, with a row of five externally tangent circles, then four on top of that, etc, such that each circle is externally tangent to its neighbors. An equilateral triangle is circumscribed around this arrangement of circles. Find the side length of the triangle.



Problem 44 (Putnam):

Rectangle $ABCD$ is inscribed in a unit circle. A point P is chosen on arc AB such that $PA = PB$. If the area of $\triangle PAB$ is equal to the area of $ABCD$, find length BC.

Problem 45:

Find all real values of x satisfying $\sqrt{2x+1} + \sqrt{6x-3} = \sqrt{3x+6} + \sqrt{5x-8}$.

Problem 46:

If $x = \sqrt[3]{7}$, find the value of $(x+1)(x^2 - x + 1)$.

Problem 47:

Blind Bill reaches into a drawer, with 11 red socks, 10 blue socks, 1 green sock, 1 yellow sock, and 15 white socks. What is the minimum number of socks he must pull out to ensure that he has at least two pairs of matching socks, with the two pairs representing different colors?

Problem 48 (Omitted):

A certain solid is made up of 12 regular pentagonal faces, 20 equilateral triangular faces, and 30 square faces. Each vertex is the meeting of one pentagonal face, one triangular face, and one square face. Find the ordered pair (e,v), where e is the number of edges of the solid, and v is the number of vertices.

Problem 49:

A circle is inscribed in a triangle with side lengths $6, 7, 9$. Find the area of this circle.

Problem 50 (HMMT):

In how many ways can 4 black and 4 white balls be placed into a 4×4 grid so that each row and each column contains exactly one black and one white ball?

Problem 51:

Alice and Bob play "Rock, Paper, Scissors." What is the probability that the first round in which one person wins is the third round?

Problem 52:

Find all real solutions to the cubic equation $5x^3 - 21x^2 + 21x - 5 = 0$.

Problem 53:

7002 identical candies are to be given to 100 students. Each student is to recieve at least 70 candies. In how many ways can the candies be distributed?

Problem 54 (GBML):

In parallelogram $ABCX$, choose points D and E on side AC such that $\frac{AE}{EC} = \frac{11}{5}$ and $\frac{AD}{AC} = \frac{5}{24}$. If the area of parallelogram $ABCX$ is 1, find the area of $\triangle BDE$.

Problem 55:

x and y are real numbers such that $2(x^2y + y^2x) = 3(x+1)(y+1) = 24$. Find all possible ordered pairs (x, y).

Problem 56:

Order the following quantities from smallest to largest: $\sqrt{26}, \sqrt{6} + \sqrt{7}, 5$.

Problem 57:

Let (a, b, c) denote the area of a triangle with side lengths a, b, c. Find the value of $\dfrac{(2007, 1337, 3000)}{(2674, 6000, 4014)}$.

Problem 58:

The base ten integer 13371337 is converted to base 9. What will its units' digit be?

Problem 59 (ARML 2007):

Find the number of points pairs of points $A = (x_1, y_1), B = (x_2, y_2)$, with $x > 0$ and an integer, such that the slope of AB is 2007, and A is to the left and below B.
EDIT: And both points are on $y = x^2$

Problem 60 (ARML 2005):

A circle with center P and radius 4 is internally tangent to a circle with center O and radius 10. OP is extended to Q such that Q is on the circle with center O. The tangent to the circle with center P from Q is drawn, and let the point of tangency be T. Extend TP to hit circle O at A and B. Find $TA \cdot TB$.

Problem 61:

Awesomemath's Tuition fee in 2007 was 2995 dollars, and in 2008, it will be 3295 dollars. What percent increase does this represent? Express your answer as a mixed number.

Problem 62:

$ABCD$ is a square, and CDE is an equilateral triangle. Find the sum of the possible measures of $\angle BEC$.

Problem 63 (Skipped):

This problem was not posted due to lack of time.

Problem 64 (MML):

Mixture A is 3 parts alcohol, 1 part water. Mixture B is 2 parts alcohol, 1 part water. x quarts of mixture A and y quarts of mixture B are mixed to obtain a new mixture with 5 parts alcohol and 2 parts water. The new mixture must have at least 24 gallons, and x and y are integers. If $x + y$ is minimized, find the ordered pair (x, y).

Problem 65 (USAMTS):

$ABCD$ is a quadrilateral with $AB = CD, \angle ABC = 77$, and $\angle DCB = 150$. The perpendicular bisectors of AD and BC meet at a point P. Find $\angle BPC$.

Problem 66:

20 people in Carl's math test take a Complex Analysis exam, not including Carl, since he is away at BROOsM. Their average score is 16. After Carl comes back and takes the exam, the average score in the test jumps to 19. What did Carl get on the exam?

Problem 67:

A dodecahedral (12-faced) die, with the faces labeled $1 - 12$, is rolled twice. Find the probability that at least one 7 is rolled.

Problem 68:

How many positive integer palindromes are there less than $133,337$?

Problem 69:

(Hard one) In $\triangle ABC$, M is the midpoint of BC, and $AM = BM$. Regular pentagons are erected from each of the three sides of the triangle. The sum of the areas of the pentagons with sides AB and AC is 1337. Find all possible integral values of the area of the third pentagon.

Problem 70:

a and b are real numbers such that $a + b = 7$ and $ab = 11$. Find $a^4 + b^4$.

Problem 71:

Saml and Maml have a race from point A at one corner of a 30 meter by 40 meter rectangular pool, to point B, at the opposite corner. Saml swims directly across the pool, and Maml runs around, on the edge of the pool. How much longer does Maml have to run than Saml swms?

Problem 72:

In a certain geometric sequence, the sum of the first 3 terms is 28, the sum of the first 4 is 60, and the sum of the first 5 is 124. Find the 11^{th} term of the sequence.

Problem 73:

A right cylinder has radius radius 5 and height 12. A string is wrapped around it evenly and tightly from top to bottom, such that the point where the string meets the top is directly above the point where it meets the bottom. If the string has length more than 1337, what is the minimum number of times that it was wrapped around?

Problem 74 (GBML):

Points A, B, C, D are on a circle O. Let $\widehat{PQ}$ denote the measure, in degrees of arc PQ (or, equivalently, $\angle POQ$). If $\widehat{BC} = \widehat{CD}$ and $2\widehat{AD} = \widehat{AB}$, find the ratio $\frac{AB}{BC}$, if O lies on BD.

Problem 75 (AHSME):

Let $f(x) = x^2 + 2bx + 1$ and $g(x) = 2a(x + b)$. The ordered pairs of real numbers (a, b) such that the graphs of the two functions do not intersect lie in a region in the ab-plane. Find the area of this region.

Problem 76 (CMO):

Regular pentagon $ABCDE$ is constructed externally to regular hexagon $ABFGHI$. Find the measure of $\angle CBF$.

Problem 77:

Saml and Maml play a game. Saml rolls a die, and if he rolls a 2, he wins. If not, Maml rolls. If Maml rolls a 6, he wins, and if not, Saml rolls. The process continues until a winner is determined. What is the probability that Saml wins?

Problem 78:

What is the remainder when 3^{2008} is divided by 80?

Problem 79:

A regular tetrahedron is a solid composed of 4 equilateral triangular faces and four vertices; it can be thought of as an equilateral triangular pyramid. If the all edges of a given regular tetrahedron measure 1 foot, find, in square inches, its volume.

Problem 80 (1988 AIME #1):

Without using a calculator, evaluate: $\sqrt{1 + 30 \cdot 31 \cdot 32 \cdot 33}$.

Problem 81:

The space diagonal of a cube measures 6 inches. Find the volume of the cube, in cubic feet.

Problem 82 (Based off 2005 State Round):

In a 1337 by 1337 grid, Samlmamls repeatedly writes his name, one letter per box. Starting in the top left, he writes an S, then moves to the right one box and writes an A, etc. When he reaches the final L, he goes to the next box and writes an S, starting over. When he reaches the end of the row, he goes to the leftmost square of the next row, until every box is filled. What is the last letter that he writes?

Problem 83 (MML):

Let $t_k = k^{-1} + k^{-2} + \ldots$, for all positive integers $k > 1$. Find the least possible k such that $t_2 + t_3 + \ldots + t_k > 3$.

Extension:

Prove that for all positive integers i, there exists some k such that $t_1 + \cdots + t_k > i$.

Problem 84 (MML):

Perpendicular chords AB and CD intersect at P in circle O such that $AP = 12, PB = 28, CP = 14$. Find $AD - PO$.

Problem 85:

Three points are chosen on a circle. What is the probability that the triangle they form is acute?

Challenge:

Generalize this to n-dimensions (it may take some linear algebra to fully rigorize this, but as a hint, use the same approach of the alternate solution).

Problem 86:

Two 12-sided die are rolled. What is the probability that the product is either one more or one less than a multiple of 3?

Problem 87:

Around a circle, a square and an equilateral triangle are circumscribed. Find the ratio of their areas.

Problem 88:

In how many ways can a 1-by-7 grid be tiled, without overlap and without empty squares, with 1-by-1 squares and 1-by-2 dominoes?

Problem 89:

At Guamo Bay, 50 prisoners are housed. Each goes in with a number on their head, with one person with the number 1, one with 2, etc. When they arrive, they are each thrown into one of 3 torture chambers with equal probability. These torture chambers are so intense that they change the number on their heads. Torture chamber #1 adds 1 to it, Torture chamber #2 divides it by 2, and torture chamber #3 multiplies it by 2, then subtracts 1.

Later, each prisoner calculates the difference between their original number and current number, and hangs the number outside the wall. What is the expected value of the sum of these values?

Problem 90 (HMMT 2008):

Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ be arbitrary nonnegative integers, with the restriction that $0 \leq a_i \leq i$ for $i = 1, 2, 3, 4$. Find the number of ordered octuples $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ such that $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

Problem 91 (Mandelbrot):

For how many positive integers n is $9 < \sqrt{n} < 10$?

Problem 92:

In calculus, if $f(x)$ is a function whose graph lies completely above the x-axis for $a \le x \le b$, then $\int_a^b f(x)dx$ is defined as the area bounded by $f(x)$, the x-axis, and vertical lines through a and b on the x-axis. Find $\int_3^7 (2x-5)dx$.

Problem 93:

One leg of a right triangle with integer side lengths has length 6. How many possible lengths are there for the hypotenuse?

Problem 94:

In a row of 20 seats, four must be painted yellow so that no two adjacent seats are yellow. In how many ways can this be done?

Problem 95:

A long fence has an opening with length 10 feet in the middle. The endpoints of this opening are A and B. The point C is 20 feet away from B such that BC and BA are perpendicular. Saml is attached to a leash, which is attached to point C. If the leash is 20 feet long, what is the area, in square feet, of the region in which Saml can move?